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Capillary–viscous forcing of surface waves

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The linear excitation of straight-crested, capillary–gravity waves on the surface ($x > 0$) of a deep, viscous liquid in response to the sinusoidal, vertical motion of a hydrophilic wall at $x = 0$ is calculated on the assumptions that: (i) the dynamical variation of the contact angle is proportional to (but not necessarily in phase with) the velocity of the contact line relative to the wall; (ii) the relative tangential velocity (slip) of the fluid below the contact line is proportional to the shear at the wall; (iii) $k_0 l_\nu \ll 1$ and $k_0 l_c = O(1)$, where k_0 is the wavenumber, l_ν is the boundary-layer thickness, and l_c is the capillary length. The contact-angle and slip coefficients are complex functions of frequency that are found to be linearly related. Physical considerations suggest that the slip length l_s ($\equiv$ slip velocity $\div$ shear at wall) should be small compared with l_ν, which, in turn, implies that the motion of the contact line must be small in that parametric domain in which linearization provides a viable description of the wave motion near the wall; however, the analysis proceeds from (i) and (ii), *qua* phenomenological hypotheses, without *a priori* restrictions on the contact-line and slip coefficients. The present results include those of Wilson & Jones (1973), who assume that the amplitude and phase of the wave slope at the wall are prescribed, and those of Hocking (1987*a*), who assumes that the variation of the wave slope at the wall is in phase with the contact-line velocity and neglects viscosity. They also include a correction for the dynamical effects of the static meniscus, which is necessarily present for any static contact angle other than $\frac{1}{2}\pi$ but is neglected in the previous analyses, and have counterparts for the closely related problem (cf. Hocking 1987*b*) of the reflection of a plane wave from a stationary wall.

1. Introduction

I consider here the excitation of a straight-crested, capillary–gravity wave with the asymptotic form

$$\eta \sim \alpha a \exp[\mathrm{i}(\omega t - k_* x)] \quad (x \to \infty) \tag{1.1}$$

in a deep, viscous liquid in response to the prescribed, vertical velocity

$$v_0 = \mathrm{i}\omega a\, \mathrm{e}^{\mathrm{i}\omega t} \tag{1.2}$$

of a hydrophilic wall at $x = 0$, where: η is the free-surface displacement; only the real parts of variables that include the factor $\exp(\mathrm{i}\omega t)$ are to be retained in the physical interpretation; a is a prescribed amplitude, which may be chosen to be real and positive; α is a dimensionless complex amplitude, which is to be determined; ω is the prescribed frequency; k_* is a complex wavenumber, the real/imaginary part of which is positive/negative (see (2.9)). The relevant lengthscales are a, $1/k_0$, l_c, l_ν, and y_c, where k_0 is determined by the dispersion relation

$$\omega^2 = gk_0 + Tk_0^3 \equiv \omega_0^2(k_0), \tag{1.3}$$

$$l_c \equiv (T/g)^{\frac{1}{2}}, \quad l_\nu \equiv (2\nu/\omega)^{\frac{1}{2}}, \tag{1.4a, b}$$

ρT is the surface tension, ν is the kinematic viscosity ($1/k_0 = 1$ cm, $l_c = 0.3$ cm, and $l_\nu = 0.03$ cm for a 5 Hz wave on clean water), and y_c is the static elevation of the contact line (the height of the meniscus) at the wall. I assume that $k_0 a$ is sufficiently small to permit linearization of the equations of motion and the boundary conditions and that

$$\epsilon \equiv k_0 l_\nu \ll 1, \quad \kappa \equiv k_0 l_c = O(1), \quad \mu \equiv k_0 |y_c| \ll 1. \qquad (1.5a\text{–}c)$$

The configuration just described, which is arguably the simplest model of the contact-line problem for surface waves, has been considered by Jewell (1967), Smith (1968), Wilson & Jones (1973), and Hocking (1987*a*) for $\mu = 0$ (flat static surface). Jewell incorporates both capillarity and viscosity and constructs the governing integral equations without invoking $\epsilon \ll 1$, but he does not report any solutions. Smith (1968) neglects capillarity, which implies that $l_c \ll l_\nu$ and is unrealistic for laboratory-scale water waves. Wilson & Jones (1973) allege that Smith's solution for $\kappa = 0$ is in error and obtain a solution for $\kappa = O(1)$ on the assumption that the amplitude and phase of the wave slope η_x are prescribed at the wall.

Hocking (1987*a*) neglects viscosity and posits the contact-line condition

$$c\eta_x = \eta_t - v_0 \quad (x = 0), \qquad (1.6)$$

where c is a parameter with the dimensions of velocity. He assumes that c is real and interprets it as the inverse local slope of contact angle *vs.* velocity in the absence of hysteresis, but this interpretation is, in my view, directly significant only for uniform motion (cf. Ablett 1923 and Ngan & Dussan V. 1989). I regard (1.6) as the simplest, non-trivial, linear hypothesis for the determination of unsteady contact-line motion and regard c as a phenomenological parameter that can be determined by laboratory measurement and that must be a complex function of ω for harmonic motion, for which η_x and the relative velocity $\eta_t - v_0$ cannot be assumed to be in phase; however, I know of no measurements of c (as *defined* by (1.6)) in the parametric domain of laboratory surface waves. It should be remarked that (1.6) excludes capillary hysteresis, which is intrinsically nonlinear and which Hocking considers separately. (Young & Davis 1987 also analyse an oscillating plate with capillary hysteresis but neglect inertial forces and do not calculate the radiated wave.)

The contact-line condition (1.6) reduces to $\eta_x = 0$ in the limit $k_0 c/\omega \to \infty$ and then is compatible with the boundary condition $v_x \to 0$ as $x \to 0$ (v is the vertical velocity) in an irrotational flow, but if $c \neq 0$ (1.6) implies a non-uniform behaviour near the contact line. This non-uniformity may be partially resolved if viscosity is admitted and the conventional no-slip hypothesis for v replaced by the slip condition† (which I also regard as a phenomenological hypothesis)

$$v - v_0 = l_s v_x \quad (x = 0,\ y < y_c), \qquad (1.7)$$

where l_s is a slip length that, like c, must be expected to be a complex function of ω.

On first considering (1.7), I had thought that it would be compatible with (1.6) only if $l_s \to c/\mathrm{i}\omega$ as $y \uparrow y_c$; in fact, allowance needs to be made for the possible non-uniformity of v_x in the neighbourhood of $x = 0$, $y = y_c$. Let

$$\zeta \equiv u_y - v_x \qquad (1.8)$$

be the vorticity (u is x-component of the velocity), which presumably is single-valued. The requirement that the shear stress $\rho\nu(u_y + v_x)$ vanish at the free surface then implies

$$\zeta + 2v_x = 0 \quad (x > 0,\ y = y_m(x)), \qquad (1.9a)$$

† The joint assumptions of contact-line motion and no slip imply a singularity in the shear stress; see Ngan & Dussan V. (1989), who also discuss alternative models for the elimination of this singularity.

where $y = y_m(x)$ is the static meniscus; this contrasts with the condition inferred from $u = u_y = 0$ at the wall,

$$\zeta + v_x = 0 \quad (x = 0,\, y < y_c). \tag{1.9b}$$

It follows that either $\zeta = v_x = 0$ at $x = 0$, $y = y_c$ (as in an irrotational flow) or that v_x is non-uniform as that point is approached.† In the latter case we have $i\omega(v - v_0) = i\omega c\eta_x = cv_x = -\frac{1}{2}c\zeta$ from (1.6) and (1.9*a*), and $v - v_0 = -l_s\zeta$ from (1.7) and (1.9*b*), in consequence of which (1.6) and (1.7) are compatible for $c, l_s \neq 0$ if and only if $l_s \to \frac{1}{2}(i\omega)^{-1}c$ as $y \uparrow y_c$ or, equivalently,

$$c = 2i\omega l_0, \quad l_0 \equiv l_s(y_c). \tag{1.10a, b}$$

Appropriate, dimensionless measures of c and l_s are

$$\gamma \equiv \frac{c}{\omega l_c}, \quad \lambda = \frac{l_0}{l_\nu}, \tag{1.11a, b}$$

which may be combined with (1.5) and (1.10) to obtain

$$\gamma = 2i\epsilon\kappa^{-1}\lambda. \tag{1.12}$$

Physical consideration of the slip condition (1.7) suggests that $|\lambda| \ll 1$ and hence, through (1.12), that $|\gamma| \ll 1$.‡ This implies that contact-line motion is negligible, as proposed and experimentally confirmed (for $v_0 = 0$) by Benjamin & Scott (1979) for surface waves in a rim-full container or as observed for the reflection of surface waves of sufficiently small amplitude and sufficiently high frequency from a gently sloping beach (see Mahony & Pritchard 1980 and Miles 1990). This may be true throughout that parametric domain in which (1.6) and (1.7) are useful approximations, in which case they reduce to $\eta_t = v_0$ and $v = v_0$, but it seems worthwhile, both conceptually and for comparison with experiment, to obtain an analytical solution that comprises both capillarity and viscosity and explicitly displays the dependence of the response on the parameters κ, γ and λ, without *a priori* restrictions on their magnitudes, in the limit $\epsilon \to 0$ (the boundary-layer approximation) and then to examine the joint limit $\kappa, \gamma, \lambda \to 0$.

I proceed as follows. In §2, I pose the boundary-value problem for $\mu = 0$ and, following Lamb (1932), give the asymptotic solutions for $k_0 x \to \infty$, in which limit (1.1) is realized, and for $k_0 y \downarrow -\infty$, in which limit the solution is essentially that of Stokes for an oscillating plate. In §3, I recapitulate Hocking's (1987*a*) solution of the inviscid problem for infinite depth (Hocking considers arbitrary depth). Neither §2 nor §3 contains new results, but they provide a necessary foundation for the subsequent determination of the joint effects of capillarity and viscosity.

In §4, I formulate the inviscid problem with allowance for a non-flat static surface and construct an approximation that is quantitatively valid for $|\mu| \ll 1$ and appears

† Note that v_x must be expected to be non-uniform near the intersection of a free surface and a rigid boundary in an irrotational flow (e.g. (3.8) implies $v_x \propto \tan^{-1}(x/-y)$ as $x, y \to 0$), and therefore also in a boundary-layer approximation obtained by perturbing an irrotational flow, but that (1.9*a*, *b*) are inferred directly from the viscous boundary conditions without further approximation.

‡ This contrasts strongly with the result inferred from the analysis of Ngan & Dussan V. (1989) for the uniform advance of a viscous liquid with velocity U through a narrow gap of breadth b in the joint limit $\nu U/T$, Ub/ν, $b/l_c \to 0$. Their result for $1/c \equiv -d\theta/dU$ at $U = 0$ implies $\gamma = O(T/\nu\omega l_c) = O(\kappa/\epsilon^2)$, which implies $|\gamma| \gg 1$ in the surface-wave regime. However, their assumptions manifestly differ from those on which (1.12) is based; in particular, $Ub/\nu \to 0$ implies the neglect of inertial forces in their derivation, whereas the equality $i\omega\eta_x = v_x$, which is crucial in the derivation of (1.10*a*), is otiose for uniform motion.

to be qualitatively valid for $\mu = O(1)$. I have not attempted to calculate the interaction of the meniscus with the viscous boundary layers (see below), but this interaction is at most $O(\epsilon\mu)$ and negligible in the present context.

In §5, I postulate viscous boundary layers of strength $O(\epsilon)$ at the wall and $O(\epsilon^2)$ at the free surface and calculate the complex amplitudes of the wave slope at the wall and the outgoing wave through $O(\epsilon)$. I simplify the calculation in §5 by assuming that l_s is constant, which is somewhat unrealistic (since l_s must be expected to vanish for $|y| \gg l_\nu$) but suffices for order-of-magnitude estimates. If only the lowest-order terms in the ordered limit ϵ, κ, μ, γ, $\lambda \to 0$ are retained the result for α, as defined by (1.1), reduces to

$$\alpha = -2\mathrm{i}\kappa[1+0.38\mu+\mathrm{i}\gamma+O(\gamma^2,\mu^2,\kappa\ln\kappa)]+\epsilon[1+\mathrm{i}-2\mathrm{i}\lambda+O(\epsilon,\lambda^2,\kappa\ln\kappa)], \quad (1.13)$$

in which $-2\mathrm{i}\kappa(1+0.38\mu)$, $2\kappa\gamma \equiv 2k_0 c/\omega$, $(1+\mathrm{i})\epsilon$ and $2\mathrm{i}\epsilon\lambda = 2\mathrm{i}k_0 l_s$ represent the respective effects of capillarity with a fixed contact line, contact-line motion, boundary-layer viscosity without slip, and viscous slip. The second and fourth of these may be combined through (1.12) to obtain $2\kappa\gamma - 2\mathrm{i}\epsilon\lambda = \kappa\gamma$ (but note that, owing to the assumption that l_s is constant, $2\mathrm{i}\epsilon\lambda$ presumably overestimates the effect of viscous slip); more importantly, each of $2\kappa\gamma$ and $2\mathrm{i}\epsilon\lambda$ is small compared with the $O(\epsilon)$ contribution of boundary-layer viscosity.

The viscous correction factor to Hocking's (1987*a*) inviscid result for η is found (in §5) to be $1+O(\epsilon)$, uniformly with respect to x, by virtue of which the lengthscale for η in the neighbourhood of the wall is l_c (rather than l_ν). The inviscid result, (3.8) below, may be expressed in terms of exponential integrals, but it suffices for illustrative purposes to consider the limit $\kappa \to 0$ with $x = O(l_c)$, which yields

$$\eta = (1-\mathrm{i}\gamma)^{-1} a \exp[\mathrm{i}\omega t-(x/l_c)][1+O(\epsilon,\ \mu,\ \kappa\ln\kappa)]. \quad (1.14)$$

It is worth noting that (1.14) also may be obtained by solving the linearized hydrostatic equation (in which the hydrodynamic pressure and the normal component of the viscous stress are neglected)

$$T\eta_{xx} = g\eta \quad (1.15)$$

subject to the contact-line condition (1.6) and a finiteness condition for $x \gg l_c$.

Surfactant contamination (which may be either accidental or intentional) is present in most laboratory configurations for which capillary phenomena are significant and may render the free surface approximately inextensible, in consequence of which the condition of vanishing shear stress is replaced by the condition of vanishing tangential velocity. The boundary layers at the surface and the wall then are both $O(\epsilon)$ but interact only at $O(\epsilon^2)$. I carry out the corresponding boundary-layer calculation in the Appendix. The dominant effect of the surface boundary layer is to render $k_* - k_0 = O(\epsilon)$, rather than $O(\epsilon^2)$ as in §§2 and 5.

Surface-wave excitation by an oscillating wall is closely related to surface-wave reflection from a stationary wall, which also has been considered by Hocking (1987*b*), and the present results – in particular (1.10) and the conclusion that the lengthscale for η near the wall is l_c – have direct counterparts for the reflection problem and also for standing waves in closed basins.

Finally, it should be emphasized that the present analysis is expected to be valid only for hydrophilic fluid–solid combinations – e.g. Photo-Flo (a wetting agent)-treated water on clean glass or *n*-butyl alcohol on clean Plexiglas. Capillary hysteresis and experimental irreproducibility appear to be ineluctable concomitants of hydrophobic combinations.

2. Boundary-value problem

The linearized boundary-value problem for $\mu = 0$ (flat static surface) is described by (cf. Lamb 1932, §349, after changing his signs of ϕ and ψ and separating the hydrodynamic and hydrostatic pressures)

$$u = \phi_x + \psi_y, \quad v = \phi_y - \psi_x, \quad p = -\rho\phi_t, \tag{2.1a–c}$$

$$\nabla^2\phi = 0, \quad \nu\nabla^2\psi = \psi_t, \tag{2.2a, b}$$

$$u = 0, \quad v - l_s v_x = v_0 \quad (x = 0,\ y < 0), \tag{2.3a, b}$$

$$v = \eta_t, \quad T\eta_{xx} - g\eta = -\frac{p}{\rho} + 2\nu v_y, \quad \nu(u_y + v_x) = 0 \quad (x > 0,\ y = 0), \tag{2.4a–c}$$

$$\phi \to 0, \quad \psi_y \to 0 \quad (y \downarrow -\infty), \tag{2.5a, b}$$

the contact-line condition (1.6) at $x = 0$, and the radiation condition (1.1) for $x \to \infty$; u and v are the x- and y-components of the velocity, and p is the hydrodynamic pressure.

The solution for $k_0 x \gg 1$, for which the conditions (1.6) and (2.3) at $x = 0$ may be relaxed, is given by (Lamb 1932, after letting $n = \mathrm{i}\omega$ and reversing the sign of kx in his solution)

$$[\phi, \psi] = (\mathrm{i}\omega/k)[(1 - \mathrm{i}\epsilon^2)\,\mathrm{e}^{ky}, \epsilon^2\,\mathrm{e}^{my}]\eta, \quad \eta = \alpha a\,\mathrm{e}^{\mathrm{i}(\omega t - kx)}, \tag{2.6a, b}$$

where

$$\epsilon = kl_\nu, \quad m = (k^2 + q^2)^{\frac{1}{2}}, \quad q = \left(\frac{\mathrm{i}\omega}{\nu}\right)^{\frac{1}{2}} = \frac{1+\mathrm{i}}{l_\nu}, \tag{2.7a–c}$$

and k is determined by the complex dispersion relation

$$(\omega - 2\mathrm{i}\nu k^2)^2 + 4\nu^2 k^3 m = gk + Tk^3 \equiv \omega_0^2(k). \tag{2.8}$$

Expanding (2.8) about $k = k_0$, where $k = k_0$ is determined by $\omega_0^2(k_0) = \omega^2$ (1.3), we obtain

$$k = k_0 - \frac{\mathrm{i}\epsilon^2\omega}{c_g(k_0)} + O(\epsilon^3) \equiv k_*, \quad c_g = \omega_0'(k), \tag{2.9a, b}$$

where c_g is the group velocity.

The solution for $k_0|y| \gg 1$, for which (2.4) may be relaxed, is given by

$$\phi = 0, \quad \psi = \frac{v_0\,\mathrm{e}^{-qx}}{q(1 + ql_s)} \tag{2.10a, b}$$

on the assumption that l_s is constant.

3. Hocking's inviscid solution

The inviscid ($\nu = 0$) solution satisfies Laplace's equation (2.2a), the boundary conditions

$$\phi_x = 0, \quad \eta_t - c\eta_x = v_0 \quad (x = 0), \tag{3.1a, b}$$

$$\phi_y = \eta_t, \quad \phi_t = T\eta_{xx} - g\eta \quad (y = 0), \tag{3.2a, b}$$

the null condition (2.5a), and the radiation condition (1.1).

Introducing the Fourier-cosine transforms

$$(\Phi, N) = \int_0^\infty (\phi, \eta)\cos kx\,\mathrm{d}x, \quad (\phi, \eta) = \frac{2}{\pi}\int_0^\infty (\Phi, N)\cos kx\,\mathrm{d}k, \tag{3.3a, b}$$

transforming (2.2a) and (3.2a, b), and invoking (3.1a) in the transform of ϕ_{xx}, we obtain

$$\Phi_{yy} - k^2 \Phi = 0, \tag{3.4}$$

$$\Phi_y = \mathrm{i}\omega N, \quad \mathrm{i}\omega\Phi + (g+Tk^2)N = -T\sigma \quad (y=0), \tag{3.5a, b}$$

where

$$\sigma \equiv \eta_x|_{x=0}, \tag{3.6}$$

the wave slope at the wall, is to be determined. The solution of (3.4), (3.5a, b) and the null condition at $y = -\infty$ (2.5a) is given by

$$\Phi = \mathrm{i}\omega k^{-1} N \mathrm{e}^{ky}, \quad N = \frac{Tk\sigma}{\omega^2 - \omega_0^2(k)}, \tag{3.7a, b}$$

where $\omega_0^2(k) = gk + Tk^3$, as in (1.3). Substituting (3.7) into (3.3b), we obtain

$$(\phi, \eta) = \frac{2}{\pi} T\sigma \int_0^\infty (\mathrm{i}\omega\, \mathrm{e}^{ky}, k) \frac{\cos kx \, \mathrm{d}k}{\omega^2 - \omega_0^2(k)}, \tag{3.8}$$

where the path of integration passes over the pole at $k = k_0(\omega = \omega_0)$ in order to satisfy (1.1). (Alternatively, k may be confined to the real axis and the Fourier integrals interpreted as Cauchy principal values, in which case a multiple of the delta function $\delta(k-k_0)$ must be added to the right-hand side of (3.7b); cf. Hocking 1987a, b.)

We determine σ by substituting η, η_x and v_0 from (3.8), (3.6) and (1.2) into (3.1b). The end result, after letting $k = k_0 \xi$, and invoking (1.3), (1.4a), (1.5b) and (1.11a), is

$$\sigma = \frac{(\mathrm{i}\omega l_c)^{-1} v_0}{I + \mathrm{i}\gamma} = \frac{(a/l_c)\,\mathrm{e}^{\mathrm{i}\omega t}}{I + \mathrm{i}\gamma}, \tag{3.9}$$

where

$$I = I(\kappa) = \frac{2\kappa}{\pi} \int_0^\infty \frac{\xi \, \mathrm{d}\xi}{D(\xi)}, \quad D = (1-\xi)[1+\kappa^2(1+\xi+\xi^2)]. \tag{3.10a, b}$$

Finally, we let $x \uparrow \infty$ in (3.8), in which limit the path of integration for the exp $(\pm \mathrm{i}kx)$ component of $\cos kx$ may be deformed into $(0, \pm \mathrm{i}\infty)$ in the complex-k plane and η is dominated by the contribution of the pole at $k = k_0(\xi \equiv 1)$ to the exp $(-\mathrm{i}kx)$ component. The invocation of (3.10) and (1.1) then yields (Hocking's 1987a result after restoring a missing factor of $1+K$ and letting $KD \rightarrow \infty$ in his (4.12))

$$\alpha = \frac{2\mathrm{i}\kappa}{(1+3\kappa^2)[I(\kappa) + \mathrm{i}\gamma]}. \tag{3.11}$$

4. Inviscid solution with meniscus

The static free surface, $y = y_{\mathrm{m}}(x)$, is governed by the capillary equation

$$y = \frac{T}{gR_s} = \frac{l_c^2 y''}{(1+y'^2)^{\frac{3}{2}}}, \tag{4.1}$$

where R_s is the static radius of curvature and $y' \equiv \mathrm{d}y/\mathrm{d}x$. The solution of (4.1), subject to the null condition $y \rightarrow 0$ for $x \rightarrow \infty$, is given by (a special case of Euler's *elastica*)

$$\frac{x}{l_c} = \log\left(\frac{\tan \frac{1}{4}\chi_c}{\tan \frac{1}{4}\chi}\right) + 2(\cos \tfrac{1}{2}\chi_c - \cos \tfrac{1}{2}\chi), \quad \frac{y}{l_c} = -2\sin \tfrac{1}{2}\chi, \quad \chi_c = \theta_0 - \tfrac{1}{2}\pi, \tag{4.2a–c}$$

where $\chi = \tan^{-1}(\mathrm{d}y/\mathrm{d}x)$ varies monotonically from 0 at $x = \infty$ to χ_c at $x = 0$ and θ_0 is the static contact angle. Expanding (4.2) in powers of $\sin\frac{1}{2}\chi$, we obtain

$$y_m = y_c \, \mathrm{e}^{-x/l_c}\,[1 + O(\sin^2\tfrac{1}{2}\chi_c)], \quad y_c = -2l_c \sin\tfrac{1}{2}\chi_c \equiv \mu k_0^{-1}, \qquad (4.3a, b)$$

which provides an adequate approximation for the present investigation. We remark that $\chi_c \lessgtr 0$, and hence that $y_c \gtrless 0$, for a hydrophilic/hydrophobic surface ($\theta_0 \lessgtr \frac{1}{2}\pi$).

The presence of the meniscus alters Hocking's model (for which $y_m = 0$) in three distinct ways: (i) the linearized free-surface conditions (3.2*a*, *b*) for $\eta = y - y_m$ must be satisfied on $y = y_m$ rather than $y = 0$; (ii) the kinematical free-surface condition (3.2*a*) must include a vertical velocity induced by the slope of the meniscus; (iii) the vertical component of the dynamical capillary force, which appears as $T\eta_{xx}$ in (3.2*b*), is

$$T\left(\frac{1}{R_d} - \frac{1}{R_s}\right) \approx T(p\eta_x)_x, \quad p \equiv (1 + y_m'^2)^{-\frac{3}{2}} = \cos^3\chi, \qquad (4.4a, b)$$

where R_d is the dynamical radius of curvature. In brief, (3.2*a*, *b*) are replaced by

$$\phi_y = \eta_t + y_m'\,\phi_x, \quad \phi_t = T(p\eta_x)_x - g\eta \quad (y = y_m), \qquad (4.5a, b)$$

where the right-hand side of (4.5*a*) is the linear approximation to $\mathrm{D}\eta/\mathrm{D}t$ and that of (4.5*b*) follows from (4.4) (p is defined by (4.4*b*) throughout this section).

We satisfy Laplace's equation (2.2*a*) and the kinematic condition (3.1*a*) by posing

$$[\phi(x, y), \eta(x)] = \frac{2}{\pi}\int_0^\infty [\Phi(k)\,\mathrm{e}^{ky}, N(k)] \cos kx \, \mathrm{d}k, \qquad (4.6a)$$

where

$$[\Phi(k), N(k)] = \int_0^\infty [\phi(x, 0), \eta(x)] \cos kx \, \mathrm{d}x, \qquad (4.6b)$$

and $\Phi(k)$ now (in contrast to §3) is the Fourier-cosine transform of $\phi(x, 0)$. Multiplying (4.5*a*, *b*) through by $\cos kx$, integrating over $0 < x < \infty$, introducing

$$m(x, k) \equiv k^{-1}[\exp(ky_m) - 1], \quad q(x) \equiv 1 - p(x) \qquad (4.7a, b)$$

in order to separate out those terms that are directly transformable, and simplifying through integration by parts, we obtain (cf. (3.5*a*, *b*))

$$\mathrm{i}\omega N - k\Phi = \frac{2k}{\pi}\int_0^\infty \Phi(\hat{k})\,\hat{k}\,\mathrm{d}\hat{k}\int_0^\infty m(x, \hat{k}) \sin kx \sin \hat{k}x \, \mathrm{d}x \qquad (4.8a)$$

and

$$(g + Tk^2)N + \mathrm{i}\omega\Phi = T\left[-p_0\sigma + \frac{2k}{\pi}\int_0^\infty N(\hat{k})\,\hat{k}\,\mathrm{d}\hat{k}\int_0^\infty q(x) \sin kx \sin \hat{k}x \, \mathrm{d}x\right]$$
$$-\frac{2\mathrm{i}\omega}{\pi}\int_0^\infty \Phi(\hat{k})\,\hat{k}\,\mathrm{d}\hat{k}\int_0^\infty m(x, \hat{k}) \cos kx \cos \hat{k}x \, \mathrm{d}x, \qquad (4.8b)$$

where σ, as defined by (3.6), is to be determined, and $p_0 \equiv p(0) = \cos^3\chi_c$.

The simultaneous integral equations (4.8*a*, *b*) may be solved by iteration, starting from the first approximation (cf. (3.7))

$$N^{(1)} = (\mathrm{i}\omega)^{-1} k\Phi^{(1)} = \frac{Tp_0\sigma k}{\omega^2 - \omega_0^2(k)}, \qquad (4.9)$$

where $\omega_0^2(k)$ is defined by (1.3). The construction of the second approximation through the substitution of (4.9) into the right-hand sides of (4.8*a*, *b*) is straightforward, but the results are cumbersome, and it suffices for our purpose to assume $|\mu| \ll 1$, adopt (4.3) and the approximation

$$m(x,k) = y_c\, e^{-x/l_c}[1+O(\mu)], \tag{4.10}$$

and neglect $p_0 - 1$ and $q = O(y_c^2/l_c^2)$. The resulting second approximation is

$$N^{(2)} = N^{(1)}\left[1-\frac{2}{\pi}\omega^2\int_0^\infty \frac{\hat{k}\,d\hat{k}}{\omega^2-\omega_0^2(\hat{k})}\int_0^\infty m(x,\hat{k})\cos(k+\hat{k})\,x\,dx\right] \tag{4.11a}$$

$$= N^{(1)}\left\{1-\frac{2\omega^2 l_c y_c}{\pi}\int_0^\infty \frac{\hat{k}\,d\hat{k}}{[\omega^2-\omega_0^2(\hat{k})][1+(k+\hat{k})^2 l_c^2]}\right\}. \tag{4.11b}$$

To determine the corresponding approximations to σ and α, we require

$$\eta(0) = \frac{2}{\pi}\int_0^\infty N^{(2)}(k)\,dk = \sigma l_c(I-\mu M) \tag{4.12}$$

and $$\eta \sim -2i\lim_{k\to k_0}[(k-k_0)N^{(2)}(k)]\,e^{-ik_0x} = \left(\frac{2i\kappa}{1+3\kappa^2}\right)(1-\mu P)\,\sigma l_c\, e^{-ik_0x} \quad (k_0x\to\infty), \tag{4.13}$$

where I is given by (3.10), $\mu \equiv k_0 y_c$ is given by (4.3b),

$$M = \frac{4}{\pi^2}\kappa^2(1+\kappa^2)\int_0^\infty\int_0^\infty \frac{\xi\eta\,d\xi\,d\eta}{D(\xi)D(\eta)[1+\kappa^2(\xi+\eta)^2]},$$

$$P = \frac{2}{\pi}\kappa(1+\kappa^2)\int_0^\infty \frac{\xi\,d\xi}{D(\xi)[1+\kappa^2(1+\xi)^2]}. \tag{4.14a, b}$$

Substituting (4.12), together with $\eta_x \equiv \sigma$ and v_0 from (1.2), into (1.6) and comparing (4.13) with (1.1), we obtain

$$\sigma = \frac{(a/l_c)\,e^{i\omega t}}{I+i\gamma-\mu M} \tag{4.15}$$

and $$\alpha = \left(\frac{2i\kappa}{1+3\kappa^2}\right)\left(\frac{1-\mu P}{I+i\gamma-\mu M}\right). \tag{4.16}$$

It follows from a comparison of (4.15) and(4.16) with (3.9) and (3.11), respectively, that the effects of the meniscus are uniformly $O(\mu P, \mu M/I)$. The most important domain is $\kappa \ll 1$, in which

$$I = -1+\frac{\kappa}{\pi}(2\ln\kappa+1)+2i\kappa+O(\kappa^2), \quad P = -\tfrac{1}{2}+O(\kappa), \quad M = 0.222+O(\kappa). \tag{4.17a–c}$$

5. Boundary-layer approximation

We now return to (2.1)–(2.5), in which viscosity is admitted but the dynamical effects of the meniscus are neglected and the boundary conditions are projected onto $y = 0$. The solution may be expanded in powers of ϵ according to

$$\phi = \frac{v_0}{k_0}\sum_{n=0}^{N}\epsilon^n\phi_n(k_0x, k_0y), \quad \eta = \frac{v_0}{i\omega}\sum_{n=0}^{N}\epsilon^n\eta_n(k_0x), \tag{5.1a,b}$$

and $$\psi = \frac{v_0}{k_0}\sum_{n=1}^{N}\epsilon^n\psi_n\left(\frac{x}{l_\nu},\frac{y}{l_\nu};k_0x,k_0y\right), \tag{5.1c}$$

where v_0 and the first ($n = 0$) approximations to ϕ and η are given by (1.2) and (3.8). The stream function ψ depends on both the *fast* variables x/l_ν and y/l_ν and the *slow* variables $k_0 x$ and $k_0 y$. The substitution of (5.1) into (2.1)–(2.5) yields a sequence of subproblems for the determination of ϕ_n, η_n and ψ_n, with σ and α to be determined by the ultimate invocation of (1.1) and (1.6) (or step-by-step through the parallel expansions of σ and α in ϵ).

We consider further the truncation $N = 1$ – i.e. the first approximation to ψ and the second approximation to ϕ, for which the contribution $2\nu v_y$ to the normal stress and the contribution $2\nu\phi_{xy}$ to the shear stress $\nu(u_y + v_x)$ are negligible and (2.1)–(2.5) may be reduced to (there is no significant advantage in the separation of the $n = 0$ and $n = 1$ components or in the introduction of dimensionless variables at this level of approximation)

$$\phi_{xx} + \phi_{yy} = 0, \quad \psi_{xx} + \psi_{yy} = q^2\psi \quad \left(q = \frac{1+\mathrm{i}}{l_\nu}\right), \tag{5.2a, b}$$

$$\phi_x = -\psi_y, \quad -(1 - l_s \partial_x)\,\psi_x = v_0 - (1 - l_s \partial_x)\,\phi_y \quad (x = 0,\ y < 0), \tag{5.3a, b}$$

$$\phi_y = \mathrm{i}\omega\eta + \psi_x, \quad \mathrm{i}\omega\phi = T\eta_{xx} - g\eta, \quad \psi_{yy} - \psi_{xx} = 0 \quad (x > 0,\ y = 0), \tag{5.4a–c}$$

$$\phi \to 0, \quad \psi_y \to 0 \quad (y \downarrow -\infty), \tag{5.5a, b}$$

where, here and subsequently, $O(\epsilon^2)$ error terms are implicit.

The first approximation to ψ is determined in terms of the first approximation to ϕ (3.8) by (5.2*b*), (5.3*b*), (5.4*c*) and (5.5*b*); however, it proves more efficient to proceed to the second approximation to ϕ before determining ψ explicitly. Fourier-transforming (5.2*a*) and (5.4*a, b*), as in §3, and invoking (5.3*a*) in the transform of ϕ_{xx}, we obtain (cf. (3.4) and (3.5))

$$\Phi_{yy} - k^2\Phi = -\psi_y(0, y), \tag{5.6}$$

$$\Phi_y = \mathrm{i}\omega N + V_1, \quad \mathrm{i}\omega\Phi + (g + Tk^2)N = -T\sigma \quad (y = 0), \tag{5.7a, b}$$

where
$$V_1 \equiv \int_0^\infty \psi_x(x, 0)\cos kx\,\mathrm{d}x, \tag{5.8}$$

and σ, the wave slope at the wall, is defined by (3.6).

The solution of (5.6), (5.7*a*) and the null condition at $y = -\infty$ is given by

$$\Phi = (\mathrm{i}\omega N + V_1)\,k^{-1}\,\mathrm{e}^{ky} + \tfrac{1}{2}k^{-1}\int_{-\infty}^0 [\mathrm{e}^{-k|y-\hat{y}|} + \mathrm{e}^{k(y+\hat{y})}]\,\psi_{\hat{y}}(0, \hat{y})\,\mathrm{d}\hat{y}, \tag{5.9}$$

which may be combined with (5.8) to obtain

$$[\omega^2 - \omega_0^2(k)]N = T\sigma k + \mathrm{i}\omega\left[\int_0^\infty \psi_x(x, 0)\cos kx\,\mathrm{d}x + \int_{-\infty}^0 \psi_y(0, y)\,\mathrm{e}^{ky}\,\mathrm{d}y\right] \tag{5.10a}$$

$$= T\sigma k + \mathrm{i}\omega k\left[\int_0^\infty \psi(x, 0)\sin kx\,\mathrm{d}x - \int_{-\infty}^0 \psi(0, y)\,\mathrm{e}^{ky}\,\mathrm{d}y\right] \tag{5.10b}$$

((5.10*b*) follows from (5.10*a*) through integration by parts).

We evaluate the first integral in (5.10*b*), which represents the free-surface boundary layer, by remarking that

$$\int_0^\infty \psi_{xx}(x, 0)\sin kx\,\mathrm{d}x = k\psi(0, 0) - \kappa^2\int_0^\infty \psi(x, 0)\sin kx\,\mathrm{d}x \tag{5.11}$$

through integration by parts and that (5.2*b*) and (5.4*c*) imply $\psi_{xx}(x,0) = \frac{1}{2}q^2\psi(x,0)$. It follows that

$$\int_0^\infty \psi(x,0)\sin kx\,\mathrm{d}x = \frac{k\psi(0,0)}{k^2+\frac{1}{2}q^2}, \tag{5.12}$$

which is $O(\epsilon^2)$, and therefore negligible, relative to the second integral in (5.10*b*).

To evaluate the second integral in (5.10*b*), which represents the boundary layer at the wall, we remark that $\psi(0,y)$, as determined by the solution of (5.2*b*), (5.3*b*) with ϕ_y approximated through (3.8) therein, (5.4*c*) and (5.5*b*), comprises a surface component that depends on both of the fast variables x/l_ν and y/l_ν but is exponentially small in $|y| \gg l_\nu$ and a wall component that depends on x/l_ν and $k_0 y$ and is independent of the surface condition (5.4*c*). The contribution to the integral of the former component is negligible within the present approximation. It follows that an adequate approximation to ψ is given by the solution of (5.2*b*), (5.3*b*) with ϕ_y approximated through (3.8) therein, and (5.5*b*):

$$\psi(x,y) = \frac{\mathrm{e}^{-qx}}{q(1+l_s q)}\left[v_0 - \frac{2}{\pi}\mathrm{i}\omega T\sigma\int_0^\infty \frac{\mathrm{e}^{ky}k\,\mathrm{d}k}{\omega^2-\omega_0^2(k)}\right] \quad (|y| \gg l_\nu). \tag{5.13}$$

Substituting (5.13) into the second integral and neglecting the first integral in (5.10*b*) we obtain

$$[\omega^2-\omega_0^2(k)]N(k) = T\sigma k\left\{1-\frac{2}{\pi}\frac{\omega^2}{q(1+l_s q)}\int_0^\infty \frac{\hat{k}\,\mathrm{d}\hat{k}}{(k+\hat{k})[\omega^2-\omega_0^2(\hat{k})]}\right\} - \frac{\mathrm{i}\omega v_0}{q(1+l_s q)}. \tag{5.14}$$

To complete the second approximation, we require

$$\eta(0) = \frac{2}{\pi}\int_0^\infty N(k)\,\mathrm{d}k = \sigma l_c(I-\epsilon_* K) + \epsilon_*(\mathrm{i}\omega)^{-1}v_0 J \tag{5.15}$$

and (the asymptotic evaluation follows the penultimate sentence in §3)

$$\eta \sim \left(\frac{2\mathrm{i}}{1+3\kappa^2}\right)\left[\sigma l_c\,\kappa(1-\epsilon_* L) + \frac{\epsilon_*(1+\kappa^2)v_0}{\mathrm{i}\omega}\right]\mathrm{e}^{-\mathrm{i}k_0 x}, \tag{5.16}$$

where

$$\epsilon_* \equiv \frac{\epsilon}{1+\mathrm{i}+2\mathrm{i}\lambda}, \tag{5.17}$$

$$\left.\begin{aligned} J = \frac{2}{\pi}(1+\kappa^2)\int_0^\infty \frac{\mathrm{d}\xi}{D(\xi)}, \quad L(\kappa) = \frac{2}{\pi}(1+\kappa^2)\int_0^\infty \frac{\xi\,\mathrm{d}\xi}{(1+\xi)D(\xi)}, \\ K = \left(\frac{2}{\pi}\right)^2\kappa(1+\kappa^2)\int_0^\infty\int_0^\infty \frac{\xi\eta\,\mathrm{d}\xi\,\mathrm{d}\eta}{(\xi+\eta)D(\xi)D(\eta)}, \end{aligned}\right\} \tag{5.18a–c}$$

D and I are given by (3.10), the paths of integration are indented over the poles, and ϵ, κ and λ are defined by (1.5*a*, *b*) and (1.11*b*). Substituting (5.15), together with $\eta_x = \sigma$ and v_0 from (1.2), into (1.6), we obtain (cf. (3.9))

$$\sigma = \left[\frac{1-\epsilon_* J(\kappa)}{I(\kappa)+\mathrm{i}\gamma-\epsilon_* K(\kappa)}\right](a/l_c)\,\mathrm{e}^{\mathrm{i}\omega t}. \tag{5.19}$$

Finally, we substitute (5.17) and (5.19) into (5.16), compare the result with (1.1) to determine α, and expand in powers of ϵ to obtain

$$\alpha = \alpha_0 + \epsilon\alpha_1 + O(\epsilon^2), \tag{5.20}$$

where α_0 is given by (3.11) or, after incorporating the meniscus correction, (4.16), and

$$\alpha_1 = \frac{(1+\mathrm{i})}{(1+3\kappa^2)[1+(1+\mathrm{i})\lambda]}\left[1+\kappa^2-\kappa\left(\frac{J+L}{I+\mathrm{i}\gamma}\right)+\frac{\kappa K}{(I+\mathrm{i}\gamma)^2}\right] \tag{5.21a}$$

$$= \left[\frac{1+\mathrm{i}}{1+(1+\mathrm{i})\lambda}\right]\left[1+\left(\frac{\kappa}{I-\mathrm{i}\gamma}\right)\left(\frac{4}{\pi}\ln\kappa+3\mathrm{i}\right)+\frac{2\kappa}{\pi(I-\mathrm{i}\gamma)^2}+O(\kappa^2\ln\kappa)\right], \tag{5.21b}$$

where (5.21*b*) follows from (5.21*a*) through (4.17*a*) for I and the corresponding approximations to J, K and L. We infer from (4.16) and (5.21*b*) that capillary and viscous effects are independent if and only if $\kappa\ln\kappa$ is negligible. If only the terms of lowest order in the limit $\epsilon, \kappa, \mu, \gamma, \lambda \rightarrow 0$ are retained in (4.16) and (5.21), (5.20) reduces to (1.13).

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Appendix. Inextensible surface

If the surface is inextensible the boundary condition (2.4*c*) is replaced by

$$u = \phi_x+\psi_y = 0 \quad (x \geqslant 0,\ y=0), \tag{A 1}$$

which, by virtue of (2.3*a*), holds for $x=0$ as well as for $x>0$. Substituting the approximation (3.8) for ϕ into (A 1), we obtain

$$\psi_y = \frac{2\mathrm{i}\omega T\sigma}{\pi}\int_0^\infty \frac{k\sin kx\,\mathrm{d}k}{\omega^2-\omega_0^2(k)}[1+O(\epsilon)] \equiv u_1(x) \quad (x\geqslant 0,\ y=0). \tag{A 2}$$

The boundary-layer approximation to the solution of (2.2*b*) and(A 2) is given by (cf. (5.13))

$$\psi_S = q^{-1}u_1(x)\,\mathrm{e}^{qy} \quad (|y| \gg l_\nu). \tag{A 3}$$

Superimposing the boundary-layer solutions (5.13) and (A 3) and proceeding as in §4, we find that (5.6) and (5.7*b*) are unchanged, while (5.7*a*) is replaced by

$$\Phi_y = \mathrm{i}\omega N\left(1+\frac{k}{q}\right)+V_1 \quad (y=0), \tag{A 4}$$

where V_1 is given by (5.8). This leads to

$$[\omega^2(1+k/q)-\omega_0^2(k)]N(k) = \text{r.h.s. } (5.14)+O(\epsilon^2) \tag{A 5}$$

in place of (5.14). This, in turn, implies the replacement of $D(\xi)$ in I, J, K and L by

$$D(\xi) = (1-\xi)[1+\kappa^2(1+\xi+\xi^2)]+\tfrac{1}{2}(1-\mathrm{i})\epsilon(1+\kappa^2)\,\xi \tag{A 6}$$

and of k_* in (1.1) by

$$k_* = k_0\left[1+\tfrac{1}{2}(1-\mathrm{i})\,\epsilon\left(\frac{1+\kappa^2}{1+3\kappa^2}\right)\right]. \tag{A 7}$$

The condition (1.9*a*) does not hold for a strictly inextensible surface, but the tangential condition for unsaturated contamination may be posed in the form $u = -l_S(u_y + v_x)$, where l_S varies from ∞ for a clean surface to 0 for a saturated surface (cf. Miles 1967); accordingly, since u must vanish at $x = 0$, so also must $u_y + v_x$, and (1.9*a*) holds for any finite value of l_S.

REFERENCES

ABLETT, R. 1923 An investigation of the angle of contact between paraffin wax and water. *Phil. Mag.* **46**, 244–256.

BENJAMIN, T. B. & SCOTT, J. C. 1979 Gravity–capillary waves with edge constraints. *J. Fluid Mech.* **92**, 241–267.

HOCKING, L. M. 1987*a* Waves produced by a vertically oscillating plate. *J. Fluid Mech.* **179**, 267–281.

HOCKING, L. M. 1987*b* Reflection of capillary–gravity waves. *Wave Motion* **9**, 217–226.

JEWELL, D. A. 1967 Unsteady flow of a viscous liquid at the intersection of a solid boundary and a free surface. Ph.D. dissertation, University of California, Berkeley.

LAMB, H. 1932 *Hydrodynamics*. Cambridge University Press.

MAHONY, J. J. & PRITCHARD, W. G. 1980 Wave reflexion from beaches. *J. Fluid Mech.* **101**, 809–832.

MILES, J. W. 1967 Surface-wave damping in closed basins. *Proc. R. Soc. Lond.* A **297**, 459–475.

MILES, J. W. 1990 Wave motion in a viscous fluid of variable depth. *J. Fluid Mech.* **212**, 365–372.

NGAN, C. G. & DUSSAN V., E. B. 1989 On the dynamics of liquid spreading over solid surfaces. *J. Fluid Mech.* **209**, 191–206.

SMITH, S. H. 1968 On the creation of surface waves by viscous forces. *Q. J. Mech. Appl. Maths* **21**, 439–450.

WILSON, S. D. R. & JONES, A. F. 1973 Surface waves produced by viscous forces. *Q. J. Mech. Appl. Maths* **26**, 339–345.

YOUNG, G. W. & DAVIS, S. H. 1987 A plate oscillating across a liquid interface: effects of contact-angle hysteresis. *J. Fluid Mech.* **174**, 327–356.